

# A realization of the quantum Lorentz group

Boyka Aneva

Theory Division, CERN, 1211 Geneva 23, Switzerland

## Abstract

A realization of a deformed Lorentz algebra is considered and its irreducible representations are found; in the limit  $q \rightarrow 1$ , these are precisely the irreducible representations of the classical Lorentz group.

Since the invention of the quantum group as a pure mathematical structure [1] much progress has been made in the realization of deformations of simple Lie algebras [?], and also Lie superalgebras, and in the construction of their representations. Quantization of space-time symmetry groups (Lorentz, Poincare, the conformal group) has remained a problem for incorporating and applying the structure of quantum groups into physical systems. A deformation of the Lorentz group has been studied in [5] and a six-generator deformed Lorentz algebra has been found in [?] in terms of the chiral  $SL(2) \times SL(2)$  generators.

In this paper we consider a realization of deformed Lorentz algebra commutation relations obeyed by the generators - rotations and boosts - and construct its representations which are the exact quantum analogue of the classical Lorentz group representations.

The quantum group is generally defined as a  $q$ -deformation of the universal enveloping algebra of the underlying classical group. Thus when introducing a quantum group one should first construct a deformed associative with a non-cocommutative Hopf algebra structure; this is usually done in terms of generators obeying deformed Lie commutation relations.

We recall that the Lorentz group contains the  $SU(2)$  subgroup of rotations  $M_i, i = 1, 2, 3$ , and the boost generators  $N_i, i = 1, 2, 3$  form an irreducible  $SU(2)$  vector representation. One can form two chiral  $SL(2)$  subgroups  $I_i^L = M_i + iN_i$  and  $I_i^R = M_i - iN_i$  which act only on spinors with undotted and dotted indices. The rotation  $SU(2)$  subgroup is the diagonal in  $SL^L(2) \times SL^R(2)$ . The matrix elements of the two-dimensional fundamental representation generators (and of the conjugated one) satisfy  $\langle N_i \rangle = \mp i \langle M_i \rangle$ , the latter expressing that  $I_i^{L,R}$  vanish when acting on functions of only dotted and undotted indices respectively.

In defining a deformed quantum Lorentz group we wish to generalize the properties of the classical Lorentz group to the  $q$ -case. We shall determine the deformed commutation relations as relations imposed on the matrix elements of the fundamental representation generators acting on two-dimensional spinors with undotted indices. We assume a full analogy with the classical case, namely that the quantum Lorentz group contains the  $SU_q(2)$  subgroup, the boost operators transform under  $SU_q(2)$  as components of irreducible tensor operators, and the fundamental representation has the properties of the two-dimensional classical spinor representation.

Let  $M_\pm, M_3$  satisfy the deformed commutation relations of  $SU_q(2)$  [7]. In the deformed case the  $SU_q(2)$  generators do not transform under the irreducible tensor representation. The  $q$ -analogue of an irreducible  $SU_q(2)$  tensor operator has been defined [8, 9] as the set of  $2l + 1$

components  $T_m^l$ ,  $m = -l, \dots, l$  obeying

$$\begin{aligned} [M_3, T_m^l] &= m T_m^l, \\ [M_\pm, T_m^l]_{q^{-m/2}} &= [l \mp m]^{1/2} [l \pm m + 1]^{1/2} T_{m\pm 1}^l q^{M_3/2} \end{aligned} \quad (1)$$

where  $[A, B]_q^\alpha = AB - q^\alpha BA$ . There is an alternative definition to (1), obtained by  $q \rightarrow q^{-1}$ . Accordingly one can construct two  $q$ -vector operators  $S$  and  $T$ :

$$\begin{aligned} S_\pm &= \pm q^\mp M_\pm q^{-M_3/2} \\ S_0 &= [2]^{-1/2} (q^{-1/2} M_- M_+ - q^{1/2} M_+ M_-) \end{aligned} \quad (2)$$

and

$$\begin{aligned} T_\pm &= \pm q^\pm M_\pm q^{M_3/2} \\ T_0 &= [2]^{-1/2} (q^{1/2} M_- M_+ - q^{-1/2} M_+ M_-) \end{aligned} \quad (3)$$

We identify the operators  $-iq^{-1/4} M_+ q^{-M_3/2}$ ,  $-iq^{-1/4} M_- q^{M_3/2}$  and  $-i[M_3] q^{-M_3/2}$  with the generators  $N_+$ ,  $N_-$  and  $N_3$  respectively, of a two-dimensional  $q$ -deformed Lorentz boost transformation, expressed in  $aq$ -tensor form. We identify further the operators  $M_\pm$  in a  $q$ -tensor form, i.e.  $M_\pm \equiv q^{-1/4} M_\pm q^\mp M_3/2$ , and the operator  $M_3$  with the rotation generators of a  $q$ -deformed Lorentz transformation. The commutation relations of a deformed Lorentz algebra are imposed as the relations obeyed by the generators of the two-dimensional Lorentz transformation. Namely, the rotations  $M_\pm, M_3$  satisfy the Lie brackets of the deformed  $U_q(su(2))$ ; the commutation relations between rotations and boosts are determined as the action of the  $U_q(su(2))$  generators on the irreducible  $su_q(2)$  tensor operators (2) and (3); the commutators between the boosts are such, that in the limit  $1 \rightarrow 1$  the classical two-dimensional boost generators are recovered. The deformed Lorentz algebra has the form

$$\begin{aligned} M_+ M_- - M_- M_+ &= [2M_3], \\ M_3 M_\pm - M_\pm M_3 &= \pm M_\pm, \\ N_+ N_- - N_- N_+ &= -[2M_3] \\ N_3 N_+ q^{1/2} - q^{-1/2} N_+ N_3 &= -M_+, \\ \tilde{N}_3 N_- q^{1/2} - q^{-1/2} N_- \tilde{N}_3 &= M_-, \\ M_3 N_\pm - N_\pm M_3 &= \pm N_\pm, \\ M_+ N_- q^{-1/2} - q^{1/2} N_- M_+ &= [2] \tilde{N}_3 + (q^{1/2} - q^{-1/2}) C_2'^q, \\ M_- N_+ q^{-1/2} - q^{1/2} N_+ M_- &= -[2] N_3 + (q^{1/2} - q^{-1/2}) C_2'^q, \\ M_+ \tilde{N}_3 q^{1/2} - q^{-1/2} \tilde{N}_3 M_+ &= -N_+, \\ M_- N_3 q^{1/2} - q^{-1/2} N_3 M_- &= N_- \end{aligned} \quad (4)$$

and all other (usual) commutators vanish. The element

$$2C_2'^q = \frac{M_+ N_- q^{-1/2} - N_- M_+ q^{1/2} + M_- N_+ q^{-1/2} - N_+ M_- q^{1/2}}{q^{1/2} - q^{-1/2}} \quad (5)$$

is central in the algebra and is the quantum analogue of the second order Lorentz Casimir  $C'_2 = -M_i N_i, i = 1, 2, 3$ . In the limit  $q \rightarrow 1$  the deformed Lorentz algebra (4) contracts to the Lie algebra of the classical Lorentz group. One can show that the operators  $M_\pm, M_3, N_\pm(q^{-1}), N_3$  satisfy the  $q$ -deformed Lorentz algebra (4) with  $q$  replaced by  $q^{-1}$  and  $N \leftrightarrow \tilde{N}_3$ . Then a  $q$ -adjoint involution on the deformed Lorentz group can be defined

$$\begin{aligned} (M_\pm)^* &= M_\mp, & M_3^* &= M_3, \\ (N_\pm^* &= N_\mp(q^{-1}), & N_3^* &= N_3 \end{aligned} \quad (6)$$

The  $q$ -deformed Lie-brackets (4) and the corresponding ones with  $q$  replaced by  $q^{-1}$  and  $N_3 \leftrightarrow \tilde{N}_3$  define a six-generator quantum Lorentz group.

We note an interesting novelty, the appearance of the central element in the defining commutation relations of the deformed Lorentz algebra (16). It is a property of the deformed relations obeyed by the generators of the quantized universal enveloping algebra which is an associative algebra with a unit and with a Poincare-Birkhoff-Witt basis.

To construct the irreducible representations of the deformed Lorentz algebra we follow the classical procedure. Namely, the representations are realized in the space of the  $U_q(su(2))$  irreducible representations with canonical basis  $|j, m\rangle_q$  with  $j$  integer or half integer and  $m = -j, \dots, j$ . The action of the rotation and boost generators on the basis vectors is given by:

$$M_\pm |j, m\rangle_q = [j \mp m]^{1/2} [j \pm m + 1]^{1/2} q^{-1/4} q^{\mp m/2} |j, m \pm 1\rangle_q \quad (7)$$

$$M_3 |j, m\rangle_q = m |j, m\rangle_q \quad (8)$$

$$\begin{aligned} N_\pm |j, m\rangle_q &= \pm c_j [j \mp m]^{1/2} [j \mp m - 1]^{1/2} q^{-1/4} q^{-(j \pm m)/2} |j - 1, m \pm 1\rangle_q \\ &- a_j [j \mp m]^{1/2} [j \pm m + 1]^{1/2} q^{-1/4} q^{\mp m/2} |j, m \pm 1\rangle_q \\ &\pm c_{j+1} [j \pm m + 1]^{1/2} [j \pm m + 2]^{1/2} q^{1/4} q^{(j \mp m)/2} |j + 1, m \pm 1\rangle_q \end{aligned} \quad (9)$$

$$\begin{aligned} N_3 |j, m\rangle_q &= c_j [j - m]^{1/2} [j + m]^{1/2} q^{-m/2} |j - 1, m\rangle_q \\ &- a_j [m] q^{-m/2} |j, m\rangle_q \\ &- c_{j+1} [j + m + 1]^{1/2} [j - m + 1]^{1/2} q^{-m/2} |j + 1, m\rangle_q \end{aligned} \quad (10)$$

The action of the operator  $C_2'^q$  on the basis vectors is given by

$$C_2'^q |j, m\rangle_q = i[l_0][l_1] |j, m\rangle_q \quad (11)$$

The coefficients  $a_j, c_j$  can be determined by using the commutators between the generators  $N_+$  and  $N_-$  and  $N_\pm$ , and  $N_3$  which results in the pair of difference equations

$$(a_{j+1}[j + 2] - a_j[j])c_{j+1} = 0, \quad (12)$$

$$c_j^2[2j - 1] - a_j^2 - c_{j+1}^2[2j + 3] = 1 \quad (13)$$

We first note that since  $j \geq 0$  there is a minimal (integer or half integer) value  $j_{min} = l_0$  and hence  $j = l_0 + 1, l_0 + 2, \dots$ . Assuming that the coefficient  $c_{l_0} = 0$  we have two possibilities, either

$$c_{l_0} = 0, \quad c_{l_0+1} \neq 0, \dots, c_{l_0+n} \neq 0, \quad c_{l_0+n+1} = 0, \quad (14)$$

and the representation is finite-dimensional, or

$$c_{l_0} = 0, \quad c_j \neq 0 \quad \text{for any } j > l_0, \quad (15)$$

and the representation is infinite-dimensional. Eqs.(12, 13) for the coefficient can be easily solved and the result, being dependent on two constants  $l_0, l_1$  is

$$a_j = \frac{i[l_0][l_1]}{[j][j+1]}, \quad c_j = \frac{i}{[j]} \sqrt{\frac{([j]^2 - [l_0]^2)([j]^2 - [l_1]^2)}{[2j-1][2j+1]}} \quad (16)$$

for any  $j > l_0$ . Since  $j = l_0 + n$ , where  $n$  is a natural number, the representation will be finite-dimensional if for some  $n$

$$[l_1]^2 = [l_0 + n + 1]^2 \quad (17)$$

Due to the property of the quantity  $[A]$  the above equation is satisfied for  $\pm l_1 = \pm(l_0 + n + 1)$ . The parameter  $l_1$  is in general a complex number, but  $l_0 + n + 1$  is a real positive number, so that the representation series will terminate if, for some real  $l_1$ ,

$$|l_1| = l_0 + n + 1. \quad (18)$$

hence the spin content of the irreducible finite-dimensional representation Lorentz  $q$ -representation is determined by

$$j = l_0, l_0 + 1, \dots, |l_1| - 1, \quad m = -j, -j + 1, \dots, j. \quad (19)$$

We summarize the result: The irreducible representation of the deformed Lorentz algebra is determined by the pair  $([l_0], [l_1])$ , where  $l_0$  is a non-negative integer or half-integer real number and  $l_1$  is a complex number. The irreducible representation corresponding to a given pair  $([l_0], [l_1])$  in the  $U_q(su(2))$  canonical basis  $|j, m\rangle_q$  is given by formulae (7-11) with the coefficients (16).

If, for some natural number  $n$ ,

$$[l_1]^2 = [l_0 + n + 1]^2 \quad (20)$$

then the representation is finite-dimensional with the possible values of  $j$  and  $m$  given by (19). If

$$[l_1]^2 \neq [l_0 + n + 1]^2, \quad (21)$$

then the representation is infinite-dimensional. In the limit  $q \rightarrow 1$ , (7-11) and (16) reproduce exactly the irreducible Lorentz group [10] representations.

We now consider the conditions under which the representations of the deformed Lorentz algebra are unitary. Since the generator  $N_3$  is self- $q$ -adjoint, according to (6), then

$$\begin{aligned} \langle j, m | N_3 | j, m \rangle &= \langle j, m | N_3^* | j, m \rangle, \\ \langle j - 1, m | N_3 | j - 1, m \rangle &= \langle j - 1, m | N_3^* | j - 1, m \rangle \end{aligned} \quad (22)$$

Hence  $a_j = \bar{a}_j$  and  $c_j = -\bar{c}_j$ . From the first of the formulae (16), it follows that the condition  $a_j = \bar{a}_j$  is satisfied if either  $[l_1]$  is arbitrary and  $[l_0] = 0$ , or  $[l_0] = 0$  is arbitrary and  $i[l_1]$  is real. The second possibility with  $q$  real means that  $l_1$  should be pure imaginary

$$l_1 = i\rho, \quad (23)$$

with  $\rho$  real. The condition  $c_j = -\bar{c}_j$  for the second of the formulae (16) means that the expression under the square root must be positive, and this is obviously the case if, only

$$[j]^2 - [l_1]^2 > 0 \quad (24)$$

We have to consider two possibilities:

- (a)  $l_0 \neq 0$  and  $[l_1]$  pure imaginary, which coincides with (23).
- (b)  $[j]^2 \geq [l_1]^2$  with  $[l_1]$  real.

The latter expression has to be satisfied for all  $j$  and this is only possible if  $[l_1]^2 \leq [1]$ . Hence the possible values of  $[l_1]$  are

$$0 < |l_1| \leq 1 \quad (25)$$

The relations between  $N_{\pm}$  and their  $q$ -adjoint yield the same values for  $l_0$  and  $l_1$ .

We thus conclude: The irreducible representations of the deformed Lorentz algebra determined by the pair  $[l_0], [l_1]$  is unitary if either  $l_1$  is pure imaginary and  $l_0$  is an arbitrary non-negative integer or half-integer, or  $l_0 = 0$  and  $l_1$  is a real number in the interval  $0 < |l_1| \leq 1$ . In the limit  $q \rightarrow 1$  the corresponding representations (7-11) reproduce exactly the infinite-dimensional Lorentz group representations [10] of the principal and complementary series respectively.

To analyze the Hopf structure of the quantum lorentz group we need to generalize to the  $q$ -case the classical picture of forming two  $SL(2)$  groups from Lorentz rotations and boosts. For this purpose we consider the operators

$$\begin{aligned} I_{\pm}^L &= M_{\pm} + iN_{\pm}, \\ I_{\pm}^R &= M_{\pm} - iN_{\pm}, \\ I_3^L &= [M_3]q^{-M_3/2} + iN_3, \\ I_3^R &= [M_3]q^{-M_3/2} - iN_3, \\ \tilde{I}_3^L &= [M_3]q^{M_3/2} + i\tilde{N}_3 \\ \tilde{I}_3^R &= [M_3]q^{M_3/2} - i\tilde{N}_3 \end{aligned} \quad (26)$$

These operators satisfy the algebra

$$\begin{aligned} I_+^L I_-^L - I_-^L I_+^L &= 2(I_3^L + \tilde{I}_3^L), \\ I_3^L I_+^L q^{1/2} - q^{-1/2} I_+^L I_3^L &= 2I_+^L, \\ \tilde{I}_3^L I_-^L q^{1/2} - q^{-1/2} I_-^L \tilde{I}_3^L &= -2I_-^L, \\ \tilde{I}_3^L I_+^L q^{-1/2} - q^{1/2} I_+^L \tilde{I}_3^L &= 2I_+^L, \\ I_3^L I_-^L q^{-1/2} - q^{1/2} I_-^L I_3^L &= -2I_-^L, \\ I_+^R I_-^R - I_-^R I_+^R &= 2(I_3^R + \tilde{I}_3^R), \\ I_3^R I_+^R q^{1/2} - q^{-1/2} I_+^R I_3^R &= 2I_+^R, \\ \tilde{I}_3^R I_-^R q^{1/2} - q^{-1/2} I_-^R \tilde{I}_3^R &= -2I_-^R, \\ \tilde{I}_3^R I_+^R q^{-1/2} - q^{1/2} I_+^R \tilde{I}_3^R &= 2I_+^R, \\ I_3^R I_-^R q^{-1/2} - q^{1/2} I_-^R I_3^R &= -2I_-^R, \end{aligned} \quad (27)$$

The generators  $I_{\pm}^L, I_3^L, \tilde{I}_3^L$  simply commute with  $I_{\pm}^R, I_3^R, \tilde{I}_3^R$ . It seems that the algebra (27) has more than six generators. Due to the relations

$$\begin{aligned} 1 + \alpha \tilde{I}_3^L + \alpha \tilde{I}_3^{\tilde{L}} &= (1 - \alpha I_3^L - \alpha I_3^R)^{-1}, \\ \tilde{I}_3^L - \tilde{I}_3^R &= (1 + \alpha \tilde{I}_3^L + \alpha \tilde{I}_3^{\tilde{L}})(I_3^L - I_3^R) \end{aligned} \quad (28)$$

with  $\alpha = (q^{1/2} - q^{-1/2})/2$ , the quantum algebra (27) is, in fact, generated by two raising, two lowering and two diagonal operators.

There exists a  $q$ -adjoint involution in the algebra (27), defined as

$$\begin{aligned} (I_{\pm}^L(q))^* &= I_{\mp}^R(q^{-1}), \\ I_3^{L*} &= I_3^R, \quad \tilde{I}_3^{L*} = \tilde{I}_3^R \end{aligned} \quad (29)$$

Denoting the two-dimensional  $q$ -deformed Lorentz representations

$$\begin{aligned} |1/2, m; l_0 = 1/2, l_1 = 3/2\rangle_q &= \tau_{1/2}, \\ |1/2, m; l_0 = 1/2, l_1 = -3/2\rangle_q &= \tilde{\tau}_{1/2}, \end{aligned} \quad (30)$$

we observe that  $I_{\pm}^L, I_3^L$  (respectively  $I_{\pm}^R, I_3^R$ ) vanish when acting on  $\tau_{1/2}$  (respectively  $\tilde{\tau}_{1/2}$ ).

The algebra (27) is the  $q$ -analogue of the chiral decomposition of the classical Lorentz group, which is exactly reproduced in the limit  $q \rightarrow 1$ .

We further introduce the shifted diagonal generators

$$\begin{aligned} T_3^{L,R} &= 2 - (q^{1/2} - q^{-1/2})I_3^{L,R}, \\ T_3^{\tilde{L},R} &= 2 + (q^{1/2} - q^{-1/2})I_3^{\tilde{L},R} \end{aligned} \quad (31)$$

which does not change the structure of the algebra (27). The co-product for the deformed algebra (27) is given by which amounts to a non-cocommutative Hopf algebra structure.

$$\begin{aligned} \Delta(I_+^L) &= I_+^L \otimes 1 + T_3^L \otimes I_+^L, \\ \Delta(I_-^L) &= I_-^L \otimes \tilde{T}_3^L + 1 \otimes I_-^L, \\ \Delta(I_+^R) &= I_+^R \otimes \tilde{T}_3^R + 1 \otimes I_+^R, \\ \Delta(I_-^R) &= I_-^R \otimes 1 + T_3^L \otimes I_-^R, \\ \Delta(T_3^{L,R}) &= T_3^{L,R} \otimes T_3^{L,R}, \\ \Delta(T_3^{\tilde{L},R}) &= T_3^{\tilde{L},R} \otimes T_3^{\tilde{L},R}. \end{aligned} \quad (32)$$

To summarize, we have defined a quantized Lorentz algebra and found that every irreducible classical Lorentz group representation labelled by  $l_0$  and  $l_1$  can be  $q$ -deformed to an irreducible  $q$ -representation of the deformed Lorentz algebra labelled by  $[l_0]$  and  $[l_1]$ . The possible values of  $l_0$  and  $l_1$  are exactly the same as in the classical case.

The author is grateful for the support and hospitality of the Theory Division at CERN where most of this work was completed. Partial support of the National Foundation for Scientific Research under contract  $\Phi$ -11 is acknowledged.

## References

- [1] V.B.Drinfeld, Quantun groups, in:Proc.ICM (MCRI, Berkeley, CA, 1986)
- [2] M.Jimbo, Lett.Math.Phys.10 (1985)63
- [3] M.Jimbo, Lett.Math.Phys.11 (1985) 247
- [4] M.Jimbo, Comm.Math.Phys. 102, (1986) 537
- [5] P.Podles and S.L.Woronowicz, Comm.Math.Phys. 130 (1990) 381
- [6] O.Ogievetsky, W.B.Schmidke, J.Wess and B.Zumino, Lett.Math.Phys.23 (1991) 233
- [7] L.C.Biedenharn, Lett.Math.Phys.20 (1990) 271
- [8] V.Rittenberg and V.Scheunert, J.Math.Phys.33 (1992) 436
- [9] Yu.F.Smirnov, V.N.Tolstoy and Yu.I.Kharitonov, Sov.J.Nucl.Phys.53 (1991) 593
- [10] M.A.Naimark, Lineaar representations of the Lorentz group (Pergamon, Oxford, 1964)